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Singularities, Trapped Sets, and Cosmic Censorship in Asymptotically Flat Space-Times

A. Królak[1] and W. Rudnicki[2]

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We show that space-time is future asymptotically predictable from a regular partial Cauchy provided that singularities are causally preceded by trapped sets. Future asymptotic predictability is a formal statement of cosmic censorship in asymptotically flat space-times. A regular partial Cauchy surface means that singularities in gravitational collapse can arise only from regular initial data. Our result confirms a supposition by Hawking that singularities forced by singularity theorems cannot be naked.

1. INTRODUCTION

The purpose of this paper is to study the validity of the cosmic censorship hypothesis (Penrose, 1969) in asymptotically flat space-times. This hypothesis asserts that singularities arising from regular initial data in a physically realistic gravitational collapse cannot be visible from infinity. The main problem in any attempt to prove this hypothesis is to define in a mathematically precise language what a "singularity arising in a physically realistic gravitational collapse" means. A very interesting suggestion was put forward by Hawking (1979). His idea is that "the main, indeed, the only, reason for believing that singularities occur in gravitational collapse is the singularity theorems." Consequently, one can suppose that only singularities forced by singularity theorems are sufficiently generic.

In the most general singularity theorem (Hawking and Penrose, 1970) applied to gravitational collapse one requires the existence of a trapped surface. An important result for the characterization of gravitational collapse by trapped surfaces was proved by Schoen and Yau (1983). They showed

[1]Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-950 Warsaw, Poland.
[2]Institute of Physics, Pedagogical University, Rejtana 16A, 35-310 Rzeszów, Poland.

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that if a sufficient amount of matter is concentrated in a region of space-time bounded from all three spatial directions, then a trapped surface must necessary form. This result has been discussed by Clarke (1988) and Bizoń *et al.* (1988).

In this paper we prove that singularities arising from regular initial data and causally related to trapped surfaces cannot be naked. To prove our theorem we use global techniques of Geroch, Hawking, and Penrose. These techniques were used to prove the existence of singularities in general space-times and to demonstrate general properties of event horizons; therefore one can expect to establish by using these methods a general relationship between singularities and event horizons. This relationship is the subject of Penrose's conjecture.

In Section 2 we introduce the notion of a regular partial Cauchy surface and we discuss its properties. We also introduce the trapped set condition which is a formal statement of Hawking's idea. In Section 3 we formulate and prove our cosmic censorship theorem and we make a few remarks about its assumptions. Our notation and fundamental definitions follow those in the monograph of Hawking and Ellis (1973).

2. PRELIMINARY NOTATIONS

The following concept clarifies the idea of nonsingular initial data in asymptotically flat space-times.

Definition 1. A partial Cauchy surface S is said to be a regular partial Cauchy surface in a weakly asymptotically simple and empty space-time if the following conditions are satisfied:

1. $\overline{D^+(S, \bar{M})} \cap \lambda \neq \varnothing$ for all generators λ of J^+.
2. S has an asymptotically simple past.
3. If $H^+(S) \neq \varnothing$, then for every past-incomplete null geodesic generator γ of $H^+(S)$ there exists a point $p \in \gamma \cap H^+(S)$ such that a set $\overline{I^-(p) \cap S}$ is compact.

Condition 1 together with condition 2 means that space-time is partially asymptotically predictable from S as defined by Tipler (1976). This fact ensures that the disjoint surfaces J^+ and J^- are in the boundary of the same asymptotically flat region. Furthermore, the partial Cauchy surface S is required to be "nice" in the sense that at least some of the structure of J^+ can be predicted from initial data on S. If S were not required to be "nice," then a breakdown of prediction could arise from a bad choice of a partial Cauchy surface S and not from the formation of singularities. [For additional justification of condition 2 see Hawking and Ellis (1973), p. 316.]

Condition 3 ensures that the initial data on S which determine the formation of the singularity are set on a compact region in S; thus, the existence of a singularity is a stable property of the initial data.

By a trapped set we mean either a trapped surface or a point p such that on every future-directed null geodesic from p the expansion θ of the null geodesics from p becomes negative (i.e., the null geodesics from p are focused by the matter or curvature and start to reconverge). The point p is sometimes called a trapped point.

Definition 2. We say that the trapped set condition holds if for every future-incomplete nonspacelike geodesic λ contained in int $D(S)$, where S is a regular partial Cauchy surface, there exists a trapped set T such that $J^+(T) \cap \lambda \neq \varnothing$.

The trapped set condition specifies singularities "forced by singularity theorems." In the most general singularity theorem of Hawking and Penrose (1970) is is not shown where the singularity is located; however, in other theorems, for example, in Penrose's (1965) theorem, the singularity is to the future of the trapped surface. One can believe that this relation will hold at least for singularities arising in $D(S)$.

3. COSMIC CENSORSHIP THEOREM

The commonly accepted idea of an asymptotically flat space-time is expressed by the notion of a weakly asymptotically simple and empty space-time. A very useful condition on a weakly asymptotically simple and empty space-time, ensuring that cosmic censorship holds, is the demand that it should be future asymptotically predictable from a partial Cauchy surface.

Definition 3. A weakly asymptotically simple and empty space-time (M, g) is future asymptotically predictable from a partial Cauchy surface S if $J^+ \subset D^+(S, \bar{M})$.

In the following we first state our cosmic censorship theorem and discuss its assumptions and then provide its proof.

Theorem. A weakly asymptotically simple and empty space-time containing a regular Cauchy surface S is future asymptotically predictable from S if:

1. $R_{ab}V^aV^b \geq 0$ for every null vector V^a everywhere in $\overline{D(S)}$.
2. $K_{[a}R_{b]\ cd[e}K_{f]}K^cK^d \neq 0$ in at least one point on every null geodesic λ contained in $D(S)$, where K^a is a tangent vector to λ.
3. The trapped set condition holds.

Remarks. 1. Conditions 1 and 2 have been discussed extensively in the literature on singularity theorems (Hawking and Penrose, 1970; Hawking and Ellis, 1973), so an extended discussion will not be given here. Condition 1 may be obtained, using Einstein's equations, from an inequality on the energy-momentum tensor T_{ab} known as the weak energy condition: $T_{ab}V^aV^b \geq 0$ for any timelike vector V^a. Condition 2 essentially requires that every null geodesic feels some nonzero gravitational tidal forces at least at one point along it.

2. All the conditions of our theorem, including the trapped set condition, need only hold in the domain of dependence of the regular surface S, i.e., only in the region which is "predictable" from S. Outside the domain of dependence space-time can be completely arbitrary, even causality and energy conditions can be violated. One should always remember the warning by Hawking (1979): "once a naked singularity has occurred anything is possible."

Before we can prove our theorem we shall need the following lemmas.

Lemma 1. Let P be a point on $H^+(S)$, where S is a partial Cauchy surface. If $\dot{I}^-(p) \cap J^+(S)$ is compact, then the null geodesic generator of $H^+(S)$ through p is past-complete.

The proof of the above lemma is identical to the proof of Lemma 8.5.5 in Hawking and Ellis (1973). In the course of the proof we only need to consider to the set $\dot{I}^-(p) \cap J^+(S)$ instead of the set $H^+(S)$.

Lemma 2. Let $J_0^+ = J^+ \cap \overline{D^+(S, \bar{M})}$ $[J_0^- = J^- \cap \overline{D^-(S, M)}]$ and let K be a compact set in

$$J^-(J_0^+, \bar{M}) \cap D^+(S, \bar{M})[J^+(J_0^-, \bar{M}) \cap D^-(S, \bar{M})]$$

where S is a regular Cauchy surface. Then the intersection $\dot{J}^+(K, \bar{M}) \cap J_0^+[\dot{J}^-(K, \bar{M}) \cap J_0^-]$ is not empty.

The proof of the above lemma is similar to that of Lemma 2 in Newman and Joshi (1988). See also Lemma 6.9.3 in Hawking and Ellis (1973).

Proof of the Theorem. Suppose that space-time (M, g) is not future asymptotically predictable from S; then by Definition 1 and Proposition 2 in Tipler (1976) there exists a null geodesic generator $\eta \in H^+(S)$ which has a future endpoint on J^+. Since η is future-complete, by conditions 1 and 2 of the theorem, η must be past-incomplete, otherwise η would contain a pair of conjugate points, but this would be impossible, as η is contained in the achronal set $H^+(S)$. By Definition 1 there exists a point p on η such that the set $C = \overline{I^-(p) \cap S}$ is compact. By Lemma 1, the set $A = \dot{I}^-(p) \cap J^+(S)$ cannot be compact. Let us put a timelike vector field on M.

If every integral curve of this field which intersects the set C also intersects the set A, we have a continuous one-to-one mapping of C onto A and hence A would have to be compact. Thus, there must exist a future-inextendible timelike curve α in $I^-(p)$. Since the set $K \equiv I^-(\alpha) \cap S$ is a closed subset of the compact set C, it also must be compact. By Lemma 2, J^- intersects $\dot{J}^-(K, \bar{M})$. Thus, there exists a past-endless, past-complete geodesic generator β of $\dot{J}^-(K)$. As α has no future endpoint, β must be also future-endless. By conditions 1 and 2 of the theorem, the null geodesic β must be future-incomplete, otherwise it would contain a pair of conjugate points, but this would be impossible, as β is contained in an achronal set $\dot{I}^-(\alpha)$. By condition 3 of the theorem there exists a trapped set T such that $J^+(T) \cap \beta \neq \varnothing$.

Now suppose that T is a trapped surface $\mathcal{T}$ and suppose that $\mathcal{T} \cap I^-(\beta) = \varnothing$. Then we can vary $\mathcal{T}$ to $\mathcal{T}'$ by a small amount so that $\hat{\chi}_{ab}g^{ab}$ and ${}_2\hat{\chi}_{ab}g^{ab}$ remains negative on $\mathcal{T}'$ and $\mathcal{T}' \cap I^-(\beta) \neq \varnothing$ (${}_1\hat{\chi}_{ab}$ and ${}_2\hat{\chi}_{ab}$ are the two second fundamental forms on $\mathcal{T}$). Hence, without loss of generality we can consider the case when $\mathcal{T} \cap I^-(\beta) \neq \varnothing$. Since by construction $I^-(\beta) \subset J^-(J_0^+, \bar{M})$ $[J_0^+ = J^+ \cap D^+(S, \bar{M})]$ and $\mathcal{T}$ is compact, by Lemma 2 the intersection $J^+(\mathcal{T}, \bar{M}) \cap J_0^+$ is not empty. Thus, there exists a null geodesic generator λ of $\dot{J}^+(T)$ with future endpoint on J_0^+; consequently, λ is future-complete. Since $J^+(\mathcal{T}) \cap J^-(J_0^+, \bar{M}) \subset \operatorname{int} D(S)$ and the set $\operatorname{int} D(S)$ is globally hyperbolic and therefore causally simple, λ must have a past endpoint on $\mathcal{T}$. Therefore by definition of the closed trapped surface and condition 1 there must be a point conjugate to $\mathcal{T}$ on λ within a finite distance from $\mathcal{T}$. This is a contradiction, since λ is contained in the achronal set $\dot{J}^+(\mathcal{T})$.

Now suppose that the trapped set T is a trapped point t. Similarly as above, we can easily obtain the required contradiction if $t \in I^-(\beta)$. If $t \in J^-(\beta)$ but $t \notin I^-(\beta)$, then let $\{t_i\}$ be an infinite sequence converging to $t_i \in I^-(b)$. By Lemma 2, $\dot{J}^+(t_i, \bar{M})$ intersects J_0^+ every t_i. Thus, there exists a null geodesic generator λ_i of $\dot{J}^+(t_i)$ with future endpoint on J_0^+ which intersects t_i. By construction, t is a limit point of the sequence $\{\lambda_i\}$. By Lemma 6.21 of Hawking and Ellis (1973), through t there exists a limit curve λ_0 of the $\{\lambda_i\}$. Since all members of $\{\lambda_i\}$ are null geodesic, λ_0 is the null geodesic as well. For each t_i, the expansion θ of the future-directed geodesic congruence from t_i cannot become negative on λ_i; otherwise, there would be a point r_i conjugate to t_i along λ_i. This is impossible, since the set $\dot{J}^+(t_i)$ is achronal. However, by definition of the trapped point t, the expansion θ of the future-directed congruence from t must become negative somewhere on λ_0; therefore, by continuity, the expansion θ of the future-directed congruence from some point t_i, close enough to t, also must become negative on λ_i. This is the required contradiction which completes the proof of our theorem. ■

The above result constitutes an improvement of the result already obtained by one of the authors (Królak, 1986) in the following respects:

1. In the course of the proof of the theorem the existence of an incomplete nonspacelike geodesic needed to invoke the trapped surface condition is proved rather then assumed as in Theorem 3.1 of Królak (1986).
2. Causality need not be assumed in the whole of space-time.

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